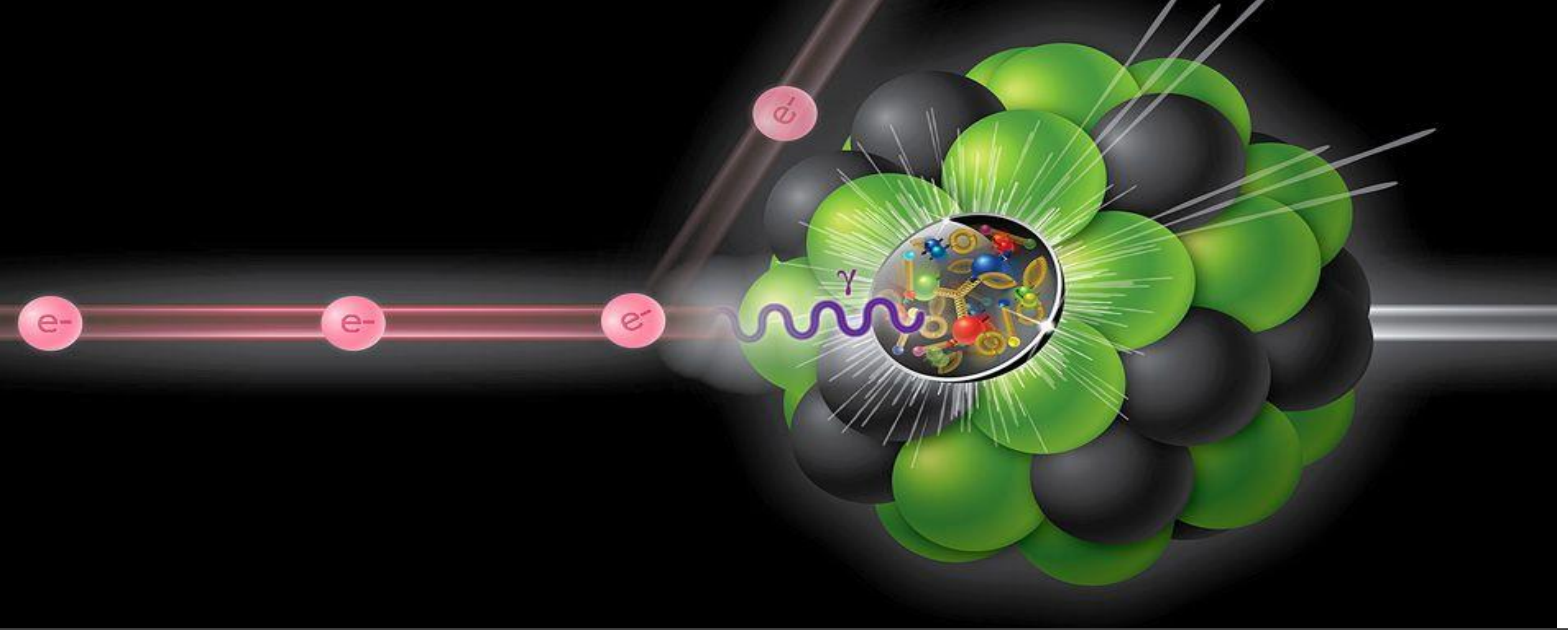

Particle Physics I

Lecture 4-5: Particle decay rates and scattering

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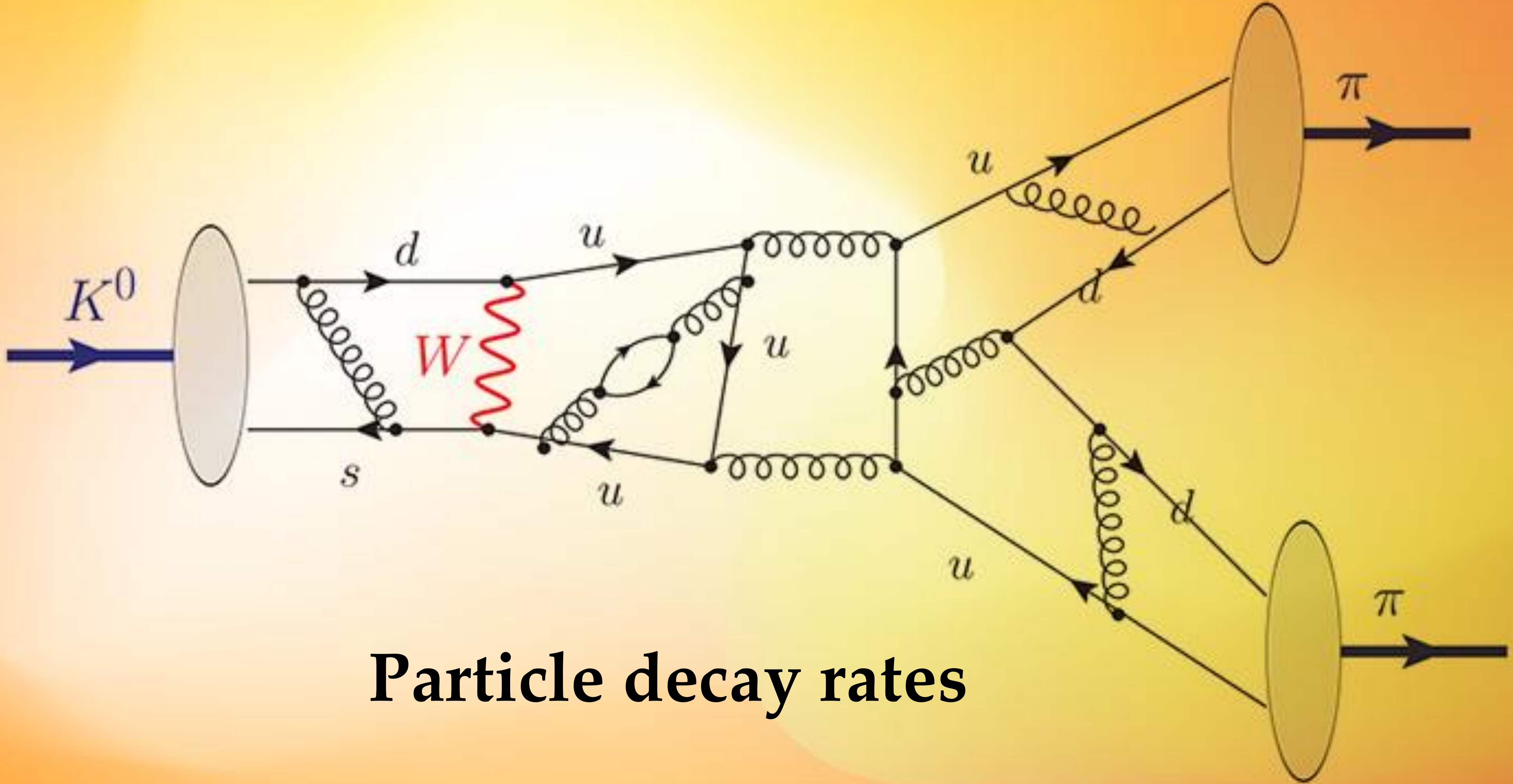
October 1st and 8th 2025

Today's learning targets

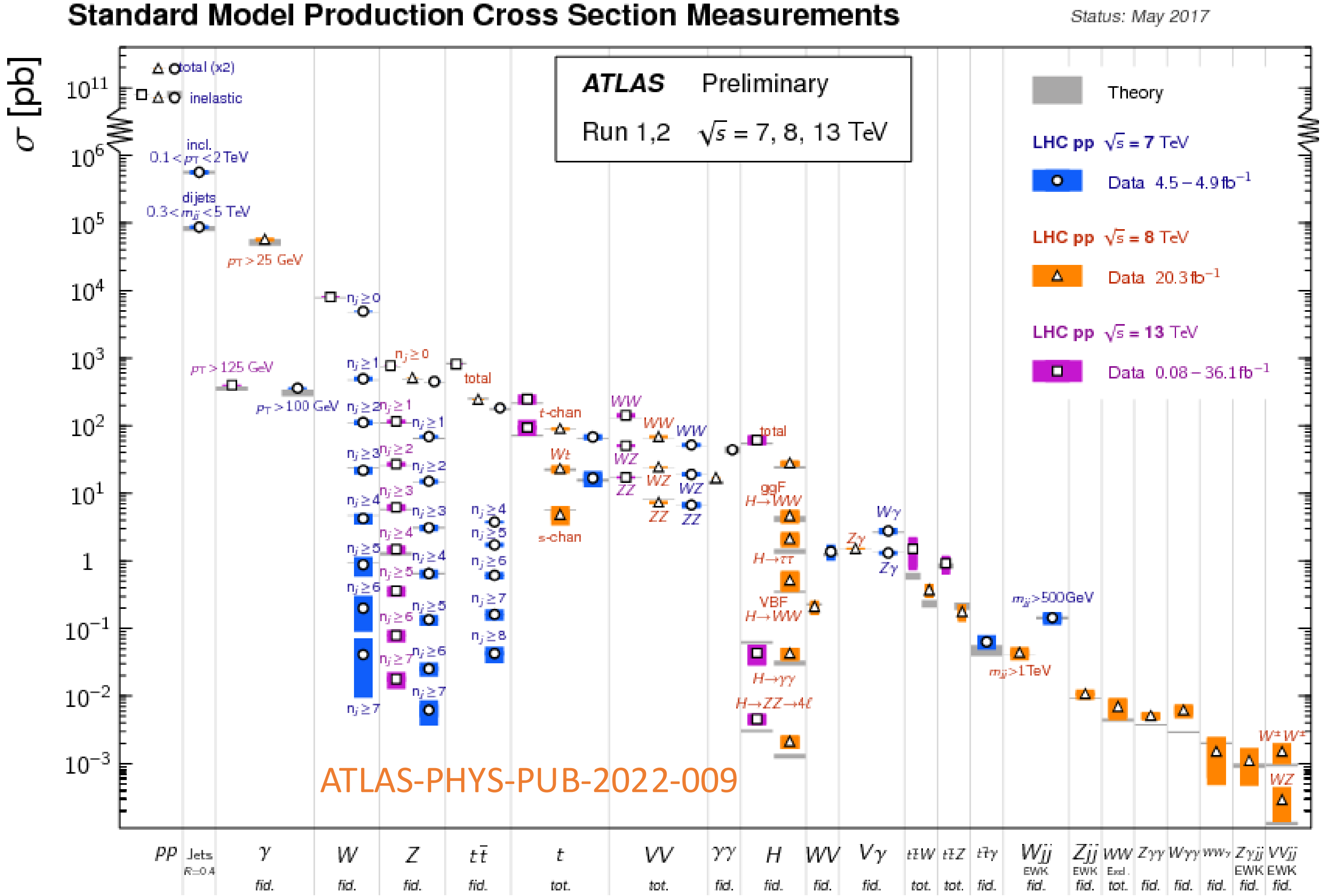
- How to compute particle decay rates
- What is Fermi's golden rule
- How to compute 2-body decay rates using Fermi's golden rule
- Relation between decay width and cross section using Fermi's Golden rule
- Calculate cross section for $2 \rightarrow 2$ scattering in the center of mass and laboratory frames



Particle decay rates

Cross sections and decay rates

- All calculation in particle physics revolve around particle **interactions** and **decays** (**transition between states**)



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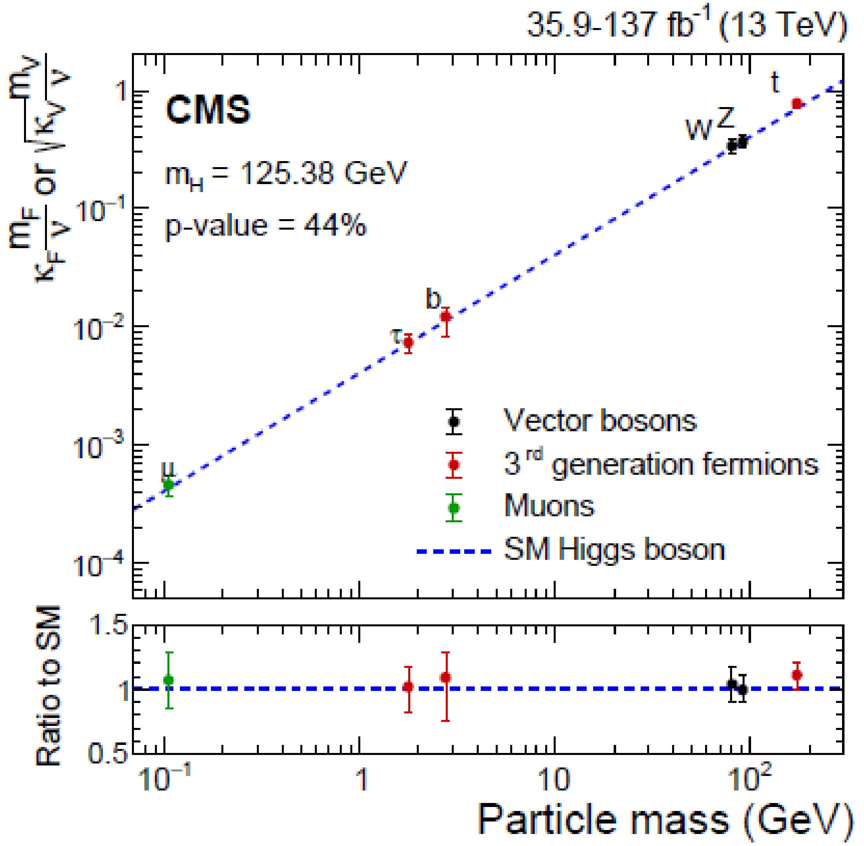

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Cross sections and decay rates

- We can calculate transition rates using Fermi's Golden Rule (see Chapter 2.3.6 in Thomson for the derivation) :

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_f) \qquad (1)$$

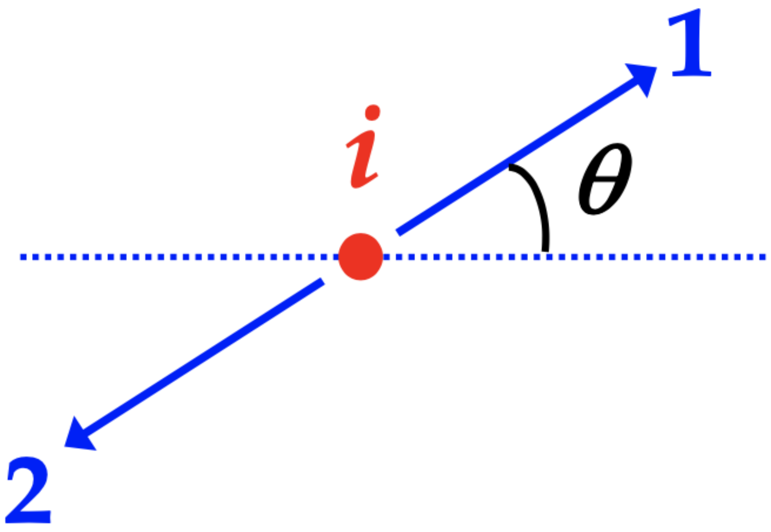
- Γ_{fi}: number of transitions per unit time from initial state $|i\rangle$ to final state $|f\rangle$ **(not Lorentz Invariant!)**
- T_{fi}: transition matrix element (ME) determined by the Hamiltonian of the interaction (j is an intermediate particle)

weak perturbation →

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ \dots \qquad (2)$$

- $\rho(E_f)$: density of final states, $\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f}$
- Decay rates depend on **matrix elements** (= fundamental particle physics) and **densities of states** (= kinematics)

Particle decay rates



- Two-body decay $i \rightarrow 1 + 2$

- The transition matrix element in first order perturbation theory is given by:

$$T_{fi} = \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_\text{i} \rangle = \int_V \Psi_1^* \Psi_2^* \hat{H} \Psi_i \, d^3x \qquad \textbf{(3)}$$

- Calculate the decay rate in first order perturbation theory describing the particle motion using plane-wave (Born approximation)

$$\Psi_1 = N e^{-i(\vec{p}.\vec{r} - Et)} = N e^{-ip.x} \qquad \textbf{(4)}$$

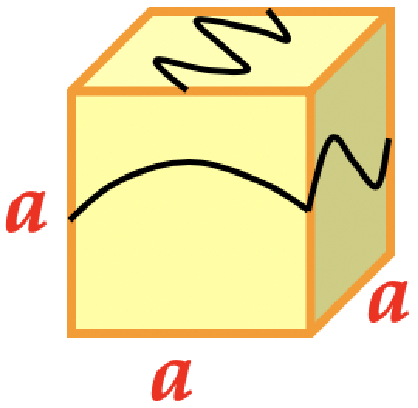
N is the normalisation

Particle decay rates

- For the decay-rate computation we need to know (in a Lorentz Invariant form)
  - wave-function normalisation
  - transition matrix element from perturbation theory
  - expression for the density of states
- Let's consider wave-function normalisation first:
  - non-relativistic formulation: normalise to one particle per cube of size a

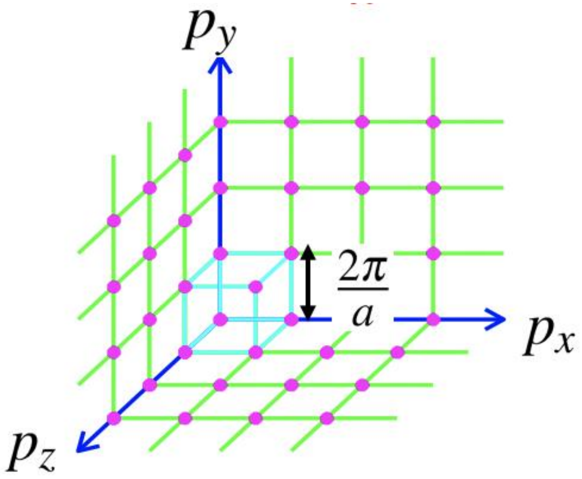
$$\int \Psi\Psi^* \, dV = N^2 a^3 = 1 \implies N^2 = 1/a^3 \qquad \textbf{(5)}$$

Non-relativistic phase space



- Apply boundary conditions: $\vec{p} = \hbar\vec{k}$
- Periodic boundary conditions on the wave function:
  - $\Psi(x+a, y, z) = \Psi(x, y, z) \implies$ quantized particle momentum

$$p_x = \frac{2\pi n_x}{a}; \qquad p_y = \frac{2\pi n_y}{a}; \qquad p_z = \frac{2\pi n_z}{a} \qquad \textbf{(6)}$$



- Volume of single state in momentum space:

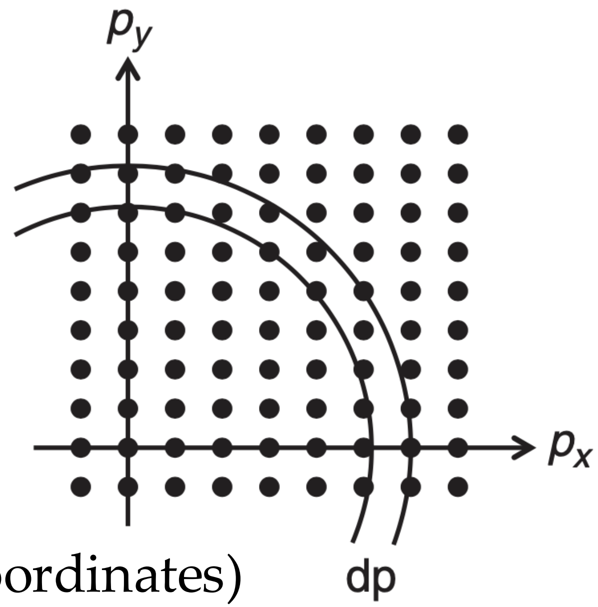
$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V} \qquad \textbf{(7)}$$

- Normalising to one particle per unit volume gives the number of states in an element

$d^3\vec{p} = dp_x dp_y dp_z$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3/V} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3} \qquad \textbf{(8)}$$

Non-relativistic phase space



- Density of states in the Golden rule: $\vec{p} = \hbar\vec{k}$

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \left|\frac{dn}{d|\vec{p}|}\frac{d|\vec{p}|}{dE}\right|_{E_f} \qquad \mathbf{(9)}$$

- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\frac{dn}{d|\vec{p}|} = \frac{1}{(2\pi)^3}\frac{d^3\vec{p}}{d|p|} = \frac{4\pi p^2 dp}{(2\pi)^3 dp} = \frac{4\pi p^2}{(2\pi)^3} \qquad \mathbf{(10)}$$

$$E^2 = p^2 + m^2 \implies 2EdE = 2pdp \implies \frac{dp}{dE} = \frac{E}{p} \approx \frac{1}{\beta} \qquad \mathbf{(11)}$$

$$\implies \rho(E_f) = \frac{4\pi p^2}{(2\pi)^3}\frac{1}{\beta} \qquad \mathbf{(12)}$$

- Larger final state momentum implies larger density of states (all other things being equal)
  - decays to lighter particles are preferred

The Golden rule revisited

$$\Gamma_{fi} = 2\pi \left|T_{fi}\right|^2 \rho(E_f) \qquad \mathbf{(13)}$$

- Rewrite the expression for density of states using a Dirac`s δ −function (backup slides?)
- Transformation of the elements using Eq. 8 and 9: $d^3\vec{p} = 4\pi p^2 dp$ (spherical coordinates)

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\delta(E - E_i)dE \quad \text{since } E_f = E_i \qquad \mathbf{(14)}$$

- Note: integrating over all final states energies but energy conservation now taken into account explicitly by the δ −function
- Hence the golden rule becomes an integral over all "allowed" final states of **any energy**:

$$\Gamma_{fi} = 2\pi \int \left|T_{fi}\right|^2 \delta(E_i - E)dn \qquad \mathbf{(15)}$$

The Golden rule revisited

- For dn in a two-body decay, we only need to consider one particle as **momentum conservation fixes the other**

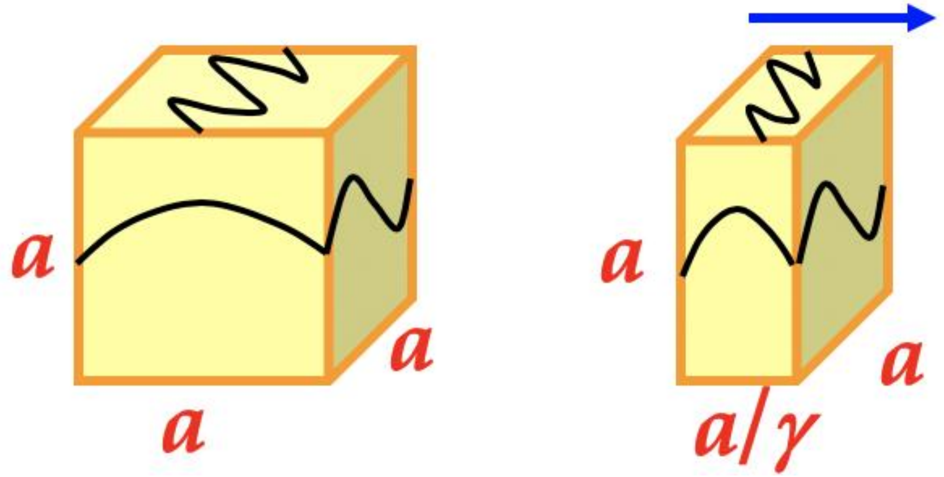
$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3\overrightarrow{p_1}}{(2\pi)^3} \qquad \textbf{(16)}$$

- We can also include momentum conservation explicitly by integrating over the momenta of both particles and using another δ −function

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{E \text{ conservation}} \underbrace{\delta^3(\overrightarrow{p_i} - \overrightarrow{p_1} - \overrightarrow{p_2})}_{\vec{p} \text{ conservation}} \underbrace{\frac{d^3\overrightarrow{p_1}}{(2\pi)^3} \frac{d^3\overrightarrow{p_2}}{(2\pi)^3}}_{\text{density of states}} \qquad \textbf{(17)}$$

Lorentz invariant phase space

- In non-relativistic QM normalise to one particle/unit volume: $\int \Psi^* \Psi dV = 1$
- Considering relativistic effects: moving to different reference frame, volume contracts by $\gamma = E/m$



- Particle density must therefore increase by γ $\Longrightarrow$ Lorentz invariant wave-function normalisation must be proportional to E particles per unit volume

Lorentz invariant phase space

- Usual convention: normalise to $2E$ particles per unit volume: $\int \Psi'^{*}\Psi' dV = 2E$
- $\Psi' = \sqrt{2E}\Psi$ is properly normalised to take into account relativistic space-time contraction
- Define Lorentz invariant matrix element, M_{fi}, in terms of the wave–functions normalized to $2E$ particles per unit volume:

$$\boldsymbol{M_{fi} = \langle \underbrace{\Psi_1' \Psi_2' \ldots}_{\text{final state}} \left| \hat{H} \right| \underbrace{\Psi_a' \Psi_b' \ldots}_{\text{initial state}} \rangle = \sqrt{2E_1 2E_2 \ldots 2E_a 2E_b} \times T_{fi}} \qquad \mathbf{(18)}$$

Two-body decay

$$M_{fi} = \langle \Psi_1' \Psi_2' | \hat{H} | \Psi_i' \rangle = \sqrt{2E_1 2E_2 2E_i} \times \langle \Psi_1 \Psi_2 | \hat{H} | \Psi_\mathrm{i} \rangle = \sqrt{2E_1 2E_2 2E_i} \times T_{fi} \qquad \mathbf{(19)}$$

- Expressing T_{fi} in terms of M_{fi} then gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int \left| M_{fi} \right|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p_i} - \vec{p_1} - \vec{p_2}) \frac{d^3\vec{p_1}}{(2\pi)^3 2E_1} \frac{d^3\vec{p_2}}{(2\pi)^3 2E_2} \qquad \mathbf{(20)}$$

- M_{fi} uses relativistically normalised wave-functions and is **Lorentz invariant**
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz invariant phase space for each final state particle
  - the factor of $2E$ arises from the wave-function normalisation

Two-body decay

- This form of Γ_{fi} is simply a rearrangement of the original equation but the integral is now frame-independent (Lorentz invariant)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle
  - this is an expected effect induced by time dilation
- Energy and momentum conservation are explicitly imposed by the δ −functions

Decay rate calculation

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\overrightarrow{p_i} - \overrightarrow{p_1} - \overrightarrow{p_2}) \frac{d^3\overrightarrow{p_1}}{(2\pi)^3 2E_1} \frac{d^3\overrightarrow{p_2}}{(2\pi)^3 2E_2} \quad \textbf{(21)}$$

- The integral is Lorentz invariant $\Longrightarrow$ can be evaluated in any frame $\Longrightarrow$ center-of-mass (CM) frame is the most convenient as the mother particle is at rest $\Longrightarrow$ $E_i = m_i, \overrightarrow{p_i} = 0$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \delta^3(\overrightarrow{p_1} + \overrightarrow{p_2}) \frac{d^3\overrightarrow{p_1}}{2E_1} \frac{d^3\overrightarrow{p_2}}{2E_2} \quad \textbf{(22)}$$

- Integrating over $\overrightarrow{p_2}$ using the delta function

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\overrightarrow{p_1}}{4E_1 E_2} \quad \textbf{(23)}$$

Decay rate calculation

- The integration over $\overrightarrow{p_2}$ using the δ −function imposes $\overrightarrow{\mathrm{p}_2} = -\overrightarrow{p_1}$ and therefore $E_2^2 = m_2^2 + |\overrightarrow{p_1}|^2$
- We can then write $d^3\overrightarrow{p_1} = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$ which leads to:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{4E_1 E_2} \qquad \mathbf{(24)}$$

⬇

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1) \delta\big(f(p_1)\big) dp_1 d\Omega \qquad \mathbf{(25)}$$

Decay rate calculation

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int \left|M_{fi}\right|^2 g(p_1)\delta\left(f(p_1)\right)\, dp_1 d\Omega \qquad \mathbf{(25)}$$

$$g(p_1) = \frac{p_1^2}{E_1 E_2} = \frac{p_1^2}{\sqrt{(m_1^2 + p_1^2)(m_2^2 + p_1^2)}} \qquad f(p_1) = m_i - \sqrt{(m_1^2 + p_1^2)} - \sqrt{(m_2^2 + p_1^2)} \qquad \mathbf{(26)}$$

- Note that $f(p_1)$ imposes energy conservation!
- CM momenta of the two decay products is fixed by $f(p_1) = 0$ for $p_1 = -p_2 = p^*$

Decay rate calculation

- Integrating Eq. 25 and using the property of the δ −function we get

$$\int g(p_1)\delta\big(f(p_1)\big)dp_1 = \frac{1}{|df/dp_1|_{p^*}}\int g(p_1)\delta(p-p^*)dp_1 = g(p^*)/\left|\frac{df}{dp_1}\right|_{p^*} \quad \mathbf{(27)}$$

- Here, p^* is the value for which $f(p^*) = 0$

$$\frac{df}{dp_1} = -\frac{p_1}{\sqrt{m_1^2+p_1^2}} - \frac{p_1}{\sqrt{m_1^2+p_1^2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2} \quad \mathbf{(28)}$$

$$\Gamma_{fi} = \frac{1}{32\pi^2E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} d\Omega = \frac{1}{32\pi^2E_i}\int |M_{fi}|^2 \left|\frac{p_1}{(E_1+E_2)}\right|_{p_1=p^*} d\Omega \quad \mathbf{(29)}$$

- From $f(p_1) = 0$ (energy conservation) we get $m_i = E_1 + E_2$

$$\boldsymbol{\Gamma_{fi} = \frac{|\vec{p^*}|}{32\pi^2E_im_i}\int |M_{fi}|^2 d\Omega} \quad \mathbf{(30)}$$

Decay rate calculation

- In the particle`s rest frame: $E_i = m_i$

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p^*}|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega \qquad (30)$$

- Valid for all two-body decays – fundamental physics contained in the matrix element, additional factors arise from the phase-space integral

- p^* can be obtained from $f(p_1) = 0$

$$m_i = \sqrt{m_1^2 + p^{*2}} + \sqrt{m_2^2 + p^{*2}}$$
$$\Rightarrow p^* = \frac{1}{2m_i^2}\sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]} \qquad (31)$$

Particle decays

- A given particle may decay to more than one decay mode

- The total decay rate per unit time Γ is the sum of the individual decay rates

$$\Gamma = \sum_j \Gamma_j \qquad \textbf{(32)}$$

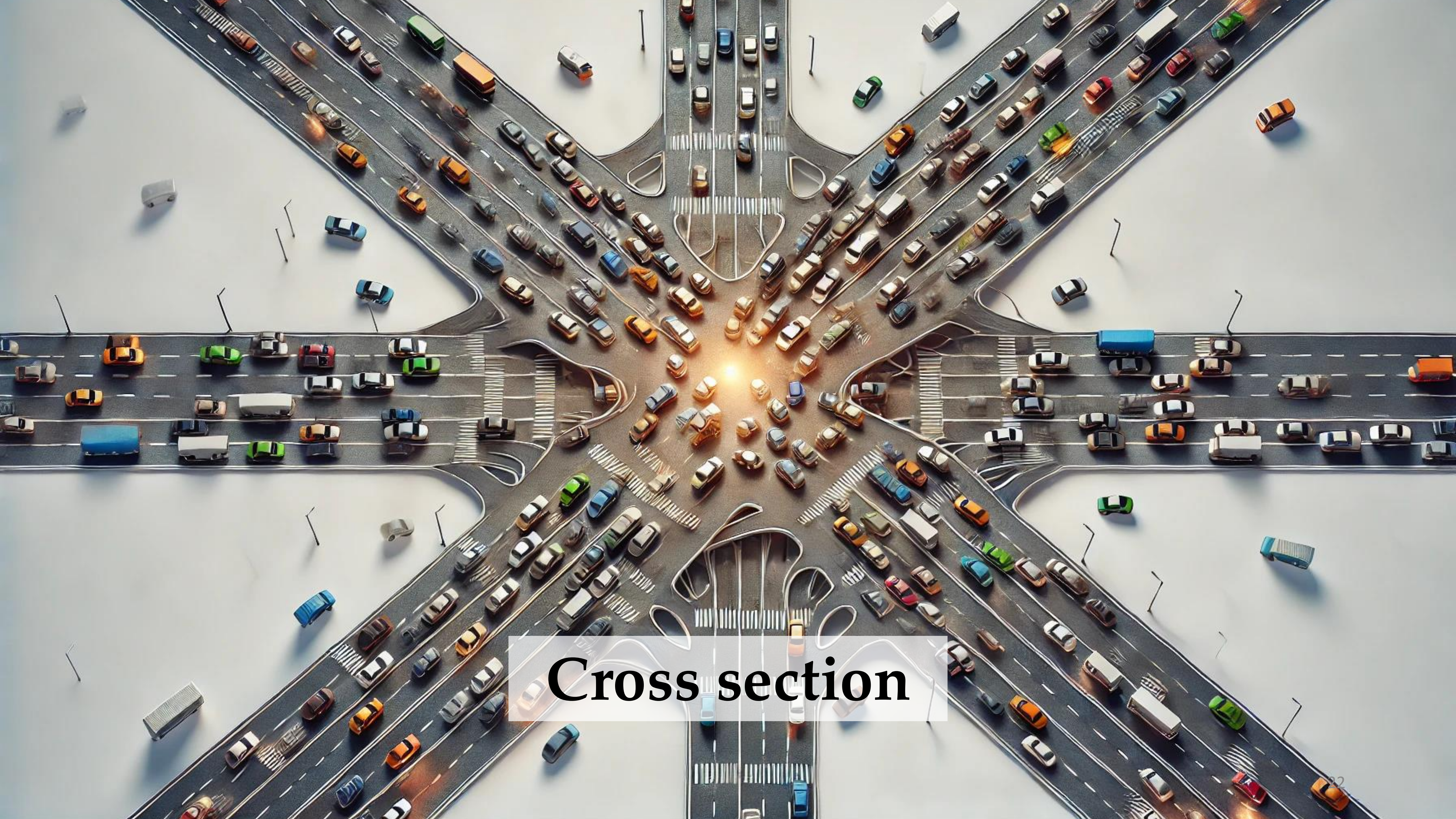
- N remaining particles after time t

$$N(t) = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau} \qquad , \tau = 1/\Gamma \qquad \textbf{(33)}$$

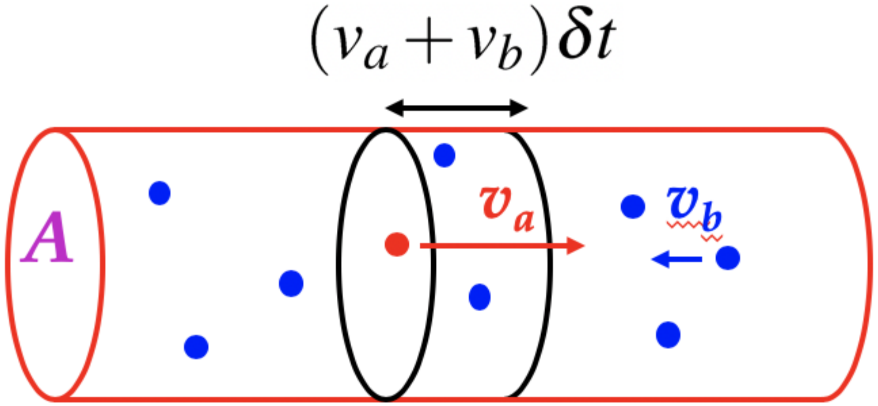
- The branching fraction for a specific decay mode is simply given by:

$$\mathrm{B(j)} = \frac{\Gamma_j}{\Gamma} \qquad \textbf{(34)}$$

Cross section



Cross section



- Consider a single particle of type a with velocity v_a, traversing a region of area A containing n_b particles of type b per unit volume

- In time δt a particle of type a traverses a region containing $\delta N = n_b(v_a + v_b)A\delta t$ particles of type b
- Interaction probability obtained from the effective cross-sectional area occupied by $n_b(v_a + v_b)A\delta t$ particles of type b:

$$\delta P = \frac{\delta N \sigma}{A} = \frac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma \qquad v = (v_a + v_b)$$

- $\Longrightarrow$ rate per particle a is $r_a = \frac{dP}{dt} = n_b v \sigma$

Cross section

- Consider a volume V, where the total reaction rate, R, is

$$R = (n_b v \sigma) \cdot (n_a V) = (n_b V)(n_a v)\sigma = N_b \phi_a \sigma$$

- **i.e. reaction rate = flux × number of target particles × cross section**

Cross section definition

$$\sigma = \frac{\text{number of interactions per unit time per target}}{\text{incident flux}}$$

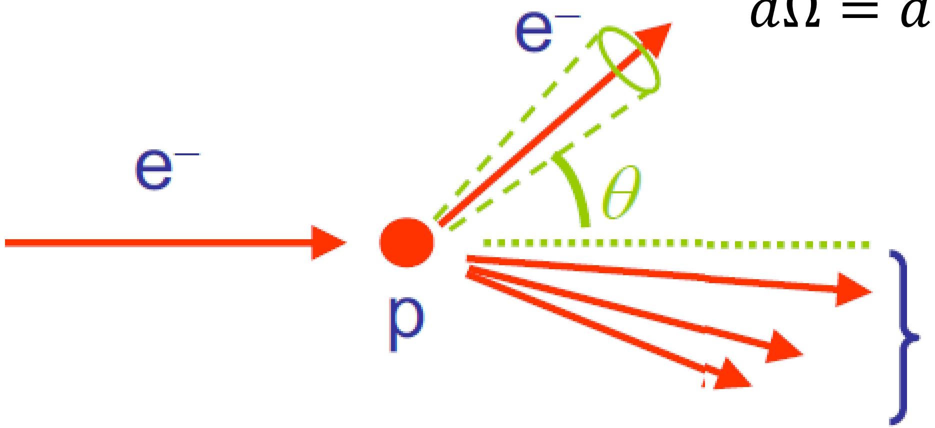
Incident flux = number of incident particles/unit area/unit time

- The “cross section”, σ, can be thought of as the effective cross-sectional area representing the size of the target object that the incoming particles must hit for the interaction to occur
- It is a measure of the probability of the interaction
- In general, this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

Differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{number of interactions per unit time per target into a solid angle } \mathrm{d}\Omega}{\text{incident flux}}$$

$$d\Omega = d(\cos\theta)d\phi$$



integrate over all other particles

$$\sigma = \int \frac{\mathrm{d}\sigma}{d\Omega} d\Omega$$

Cross section calculation

- Consider scattering process $1 + 2 \to 3 + 4$
- Start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int \left|T_{fi}\right|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\overrightarrow{p_1} + \overrightarrow{p_2} - \overrightarrow{p_3} - \overrightarrow{p_4}) \frac{d^3\overrightarrow{p_3}}{(2\pi)^3} \frac{d^3\overrightarrow{p_4}}{(2\pi)^3} \qquad \mathbf{(1)}$$

- Here T_{fi} is the transition matrix for a normalization of 1 particle per unit volume
- Rate/Volume = (flux of 1) × (number density of 2) × $\sigma = n_1(v_1 + v_2)n_2\sigma$
- For 1 target particle per unit volume, the rate is: $(v_1 + v_2)\sigma \Longrightarrow \sigma = \frac{\Gamma_{fi}}{v_1+v_2}$

$$\sigma = \frac{(2\pi)^4}{(v_1 + v_2)} \int \left|T_{fi}\right|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\overrightarrow{p_1} + \overrightarrow{p_2} - \overrightarrow{p_3} - \overrightarrow{p_4}) \frac{d^3\overrightarrow{p_3}}{(2\pi)^3} \frac{d^3\overrightarrow{p_4}}{(2\pi)^3} \qquad \mathbf{(2)}$$

These are not Lorentz-invariant terms

Cross section calculation

- To obtain Lorentz-invariant form we start by using wave functions normalised to $2E$ particles per unit volume: $\Psi' = \sqrt{2E}\,\Psi$

- Again define Lorentz-invariant matrix element $|M_{fi}| = \sqrt{2E_1 2E_2 2E_3 2E_4}|T_{fi}|$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\overrightarrow{p_1} + \overrightarrow{p_2} - \overrightarrow{p_3} - \overrightarrow{p_4}) \frac{d^3\overrightarrow{p_3}}{2E_3} \frac{d^3\overrightarrow{p_4}}{2E_4} \qquad (\mathbf{3})$$

- The integral is now written in Lorentz-invariant form

- The quantity $F = 2E_1 2E_2(v_1 + v_2)$ can be written in terms of a scalar product of 4-vectors and is also LI

$$F = 4\sqrt{(p_1^{\mu} p_{2,\mu}) - m_1^2 m_2^2} \qquad (\mathbf{4})$$

- $\Longrightarrow$ the cross section is a Lorentz invariant quantity

Two special cases of Lorentz-invariant flux

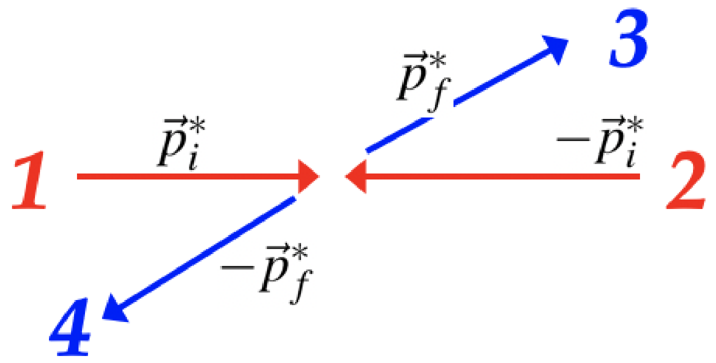
1. Center-of-Mass frame (CoM):

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2\left(\frac{p^*}{E_1}+\frac{p^*}{E_2}\right) \\ &= 4p^*(E_2+E_1) \\ &= 4p^*\sqrt{s} \end{aligned}$$

2. Target particle (particle 2) at rest:

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1m_2v_1 \\ &= \frac{4E_1m_2|\overrightarrow{p_1}|}{E_1} \\ &= 4m_2|\overrightarrow{p_1}| \end{aligned}$$

$2 \rightarrow 2$ body scattering in CoM frame



- We will apply the Lorentz-invariant formula for the interaction cross section to the most common cases
- $2 \rightarrow 2$ body scattering in the CoM frame:

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4)\delta^3(\overrightarrow{p_1} + \overrightarrow{p_2} - \overrightarrow{p_3} - \overrightarrow{p_4}) \frac{d^3\overrightarrow{p_3}}{2E_3}\frac{d^3\overrightarrow{p_4}}{2E_4} \qquad \mathbf{(5)}$$

- We can use $\overrightarrow{p_1} + \overrightarrow{p_2} = 0$ and $E_1 + E_2 = \sqrt{s}$

$$\sigma = \frac{(2\pi)^{-2}}{4|\overrightarrow{p_i}|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4)\delta^3(\overrightarrow{p_3} + \overrightarrow{p_4}) \frac{d^3\overrightarrow{p_3}}{2E_3}\frac{d^3\overrightarrow{p_4}}{2E_4} \qquad \mathbf{(6)}$$

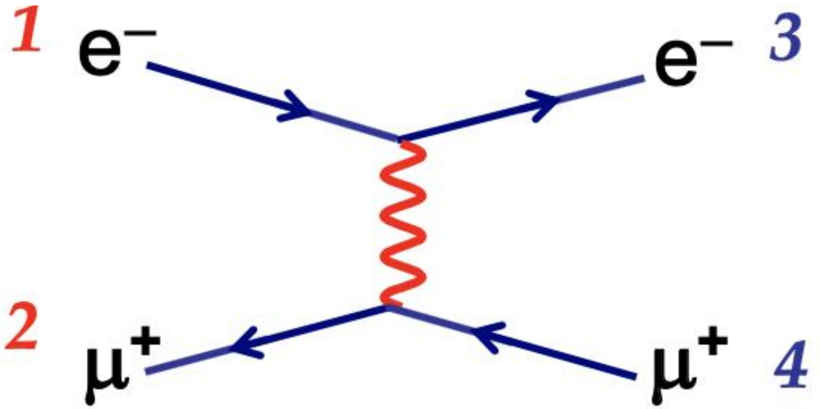
$2 \to 2$ body scattering in CoM frame

- The integral is exactly the same as in the particle decay calculation but with m_i replaced by $\sqrt{s}$
- $2 \to 2$ body scattering in the CoM frame:

$$\sigma = \frac{(2\pi)^{-2}}{4|\overrightarrow{p_i}|\sqrt{s}} \frac{|\overrightarrow{p_f}|}{4\sqrt{s}} \int |M_{fi}|^2 \, d\Omega^*$$

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\overrightarrow{p_f^*}|}{|\overrightarrow{p_i^*}|} \int |M_{fi}|^2 \, d\Omega^* \qquad \textbf{(7)}$$

$2 \to 2$ body scattering in CoM frame



- Elastic scattering: $\left|\overrightarrow{p_i^*}\right| = \left|\overrightarrow{p_f^*}\right|$

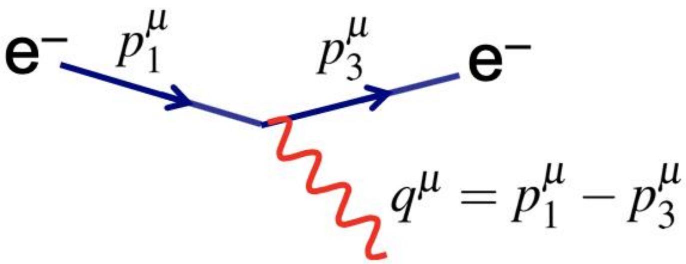
$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int \left|M_{fi}\right|^2 d\Omega^* \qquad \mathbf{(8)}$$

- For calculating the total Lorentz-invariant cross section, the result from the previous page is sufficient
  - not so useful for computing the differential cross section in a rest frame other than CoM
  - $d\Omega^* = d(cos\theta^*)d\phi^*$ refers to the angles in CoM frame

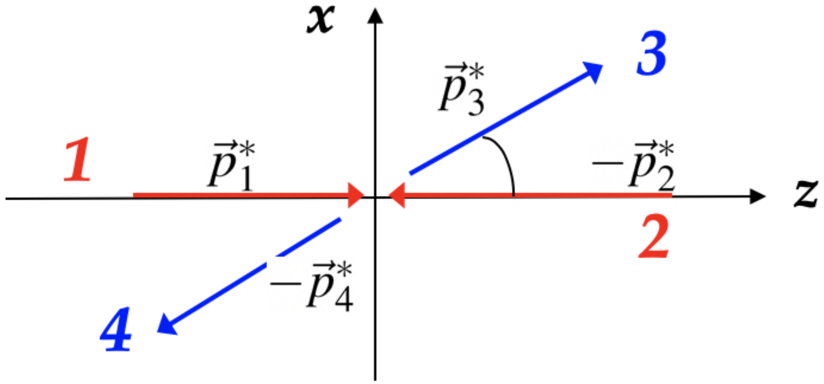
$$d\sigma = \frac{1}{64\pi^2 s} \frac{\left|\overrightarrow{p_f^*}\right|}{\left|\overrightarrow{p_i^*}\right|} \left|M_{fi}\right|^2 d\Omega^* \qquad \mathbf{(9)}$$

- We would need to find a Lorentz-invariant expression for $d\sigma$

$2 \rightarrow 2$ body scattering in CoM frame



- Express $d\Omega^*$ in terms of the Mandelstam t:
  - $t = q^2 = (p_1 - p_3)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_3$
- In CoM frame:
  - $p_1^{*\mu} = \left(E_1^*, 0,0, \left|\overrightarrow{p_1^*}\right|\right), p_3^{*\mu} = \left(E_3^*, \left|\overrightarrow{p_3^*}\right| \sin\theta^*, 0, \left|\overrightarrow{p_3^*}\right| \cos\theta^*\right)$
  - $p_1^{*\mu} \cdot p_3^{*\mu} = E_1^* E_3^* - |p_1^*||p_3^*| \cos\theta^*$
  - $t = m_1^2 + m_2^2 - 2E_1^* E_3^* + 2|p_1^*||p_3^*| \cos\theta^*$
  - $\Longrightarrow dt = 2|p_1^*||p_3^*| d(\cos\theta^*)$



- $d\Omega^* = d(\cos\theta^*) d\phi^* = \frac{dt d\phi^*}{2|p_1^*||p_3^*|}$

$$d\sigma = \frac{1}{128\pi^2 s \left|\overrightarrow{p_1^*}\right|^2} \left|M_{fi}\right|^2 dt d\phi^* \quad \textbf{(10)}$$

$2 \rightarrow 2$ body scattering in CoM frame

- Finally, integrating over $d\phi^*$(assuming no ϕ^* dependence on $\left|M_{fi}\right|^2$

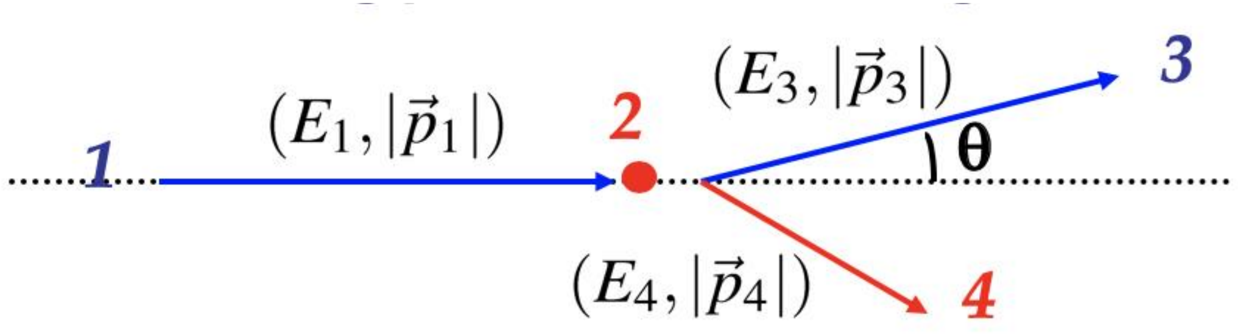
$$\frac{d\sigma}{dt} = \frac{1}{64\pi\, s\left|\overrightarrow{p_1^*}\right|^2}\left|M_{fi}\right|^2 \qquad \textbf{(11)}$$

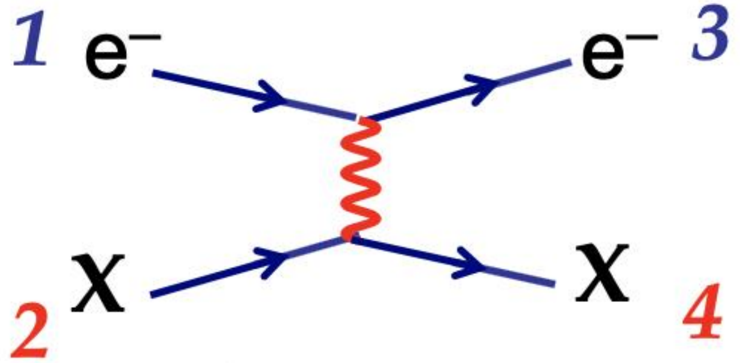
- All quantities at Lorentz-invariant and therefore it applies to any rest frame
  - $\left|\overrightarrow{p_1^*}\right|$ is constant fixed by energy-momentum conservation
  - $\left|\overrightarrow{p_1^*}\right|^2 = \frac{1}{4s}[s\ -(m_1+m_2)^2][s-(m_1-m_2)^2]$
- Example of how to use $d\sigma/dt$: consider elastic scattering in the lab. frame where we can neglect the mass of the incoming particle (e.g. electron or neutrino scattering):$\left|\overrightarrow{p_1^*}\right|^2 = (s-m_2^2)^2/(4s)$

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\left(s-m_2^2\right)^2}\left|M_{fi}\right|^2 \qquad \textbf{(12)}$$

$2 \rightarrow 2$ body scattering in laboratory frame

- The other commonly occurring case is scattering from a fixed target in a laboratory frame (e.g electron-proton scattering)
- Take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, m_2 = m_4 = M$





- Express the cross section in terms of the scattering angle of the e^-: $d\Omega = 2\pi d(\cos\theta)$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{\mathrm{d}t}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} \tag{13}$$

$2 \rightarrow 2$ body scattering in laboratory frame

- Four-momenta of the particles
  - $p_1 = (E_1, 0, 0, E_1)$
  - $p_2 = (M, 0, 0,0)$
  - $p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta)$
  - $p_4 = (E_4, \overrightarrow{p_4})$
  - $\Longrightarrow t = (p_1 - p_3)^2 = -2p_1p_3 = -2E_1E_3(1 - \cos\theta)$
- From $(E, \vec{p})$ conservation $p_1 + p_2 = p_3 + p_4$ we can express t in terms of p_2 and p_4
  - $t = (p_2 - p_4)^2 = 2M^2 - 2ME_4 = -2M(M - E_4) = -2M(E_1 - E_3)$
- E_1 is constant (the energy of the incoming particle):

$$\frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)} \qquad \mathbf{(14)}$$

$2 \rightarrow 2$ body scattering in laboratory frame

- Equating the two expressions for t we get
  - $E_3 = \frac{E_1 M}{M+E_1-E_1\cos\theta}$
  - $\frac{dE_3}{d(\cos\theta)} = \frac{E_1^2 M}{(M+E_1-E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{d\sigma}{dt} = \frac{E_3^2}{16\pi^2(s-M^2)^2}\left|M_{fi}\right|^2 \qquad \mathbf{(15)}$$

- Using $s = (p_1+p_2)^2 = M^2 + 2p_1\cdot p_2 = M^2 + 2ME_1$, as $p_1^2 = 0$, gives $(s-M^2) = 2ME_1$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2\left|M_{fi}\right|^2 \quad \text{(in the limit of } m_1 \rightarrow 0\text{)} \qquad \mathbf{(16)}$$

$\mathbf{2} \rightarrow \mathbf{2}$ body scattering in laboratory frame

- Express E_3 as a function of θ: $E_3 = \frac{ME_1}{M+E_1(1-\cos\theta)}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1(1-\cos\theta)}\right)^2 \left|M_{fi}\right|^2 \qquad \mathbf{(17)}$$

- General form of $2 \rightarrow 2$ body scattering in the lab frame in case the mass m_1 can`t be neglected

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{|\overrightarrow{p_1}|m_1}\frac{|\overrightarrow{p_3}|^2}{|\overrightarrow{p_3}|(E_1+m_2) - E_3|\overrightarrow{p_1}|\cos\theta}\left|M_{fi}\right|^2 \qquad \mathbf{(18)}$$

- There is only one independent variable: the angle θ, from conservation of energy:

$$(E_1+m_2) = \sqrt{|\overrightarrow{p_3}|^2 + m_3^2} + \sqrt{|\overrightarrow{p_1}|^2 + |\overrightarrow{p_3}|^2 - 2|\overrightarrow{p_1}||\overrightarrow{p_3}|\cos\theta + m_4^2}$$

Summary

- We used a Lorentz-invariant formulation of Fermi's golden rule to derive decay rates and cross sections
  - Expressed in the Lorentz-invariant Matrix Element (wave-functions normalised to $2E$/unit volume)
- **Particle decay width:**

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega \qquad \mathbf{(19)}$$

$p^* = \frac{1}{2m_i}\sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]}$(function of the mass of the particles)

- **Scattering cross section in CoM frame:**

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p_f}^*|}{|\vec{p_i}^*|} \int |M_{fi}|^2 \, d\Omega^* \qquad \mathbf{(20)}$$

Summary

- Invariant differential cross section valid in all frames:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi\ s\left|\overrightarrow{p_i}^*\right|^2}\int \left|M_{fi}\right|^2 \qquad \textbf{(21)}$$

$$\left|\overrightarrow{p_i^*}\right|^2 = \frac{1}{4s}[s\ -(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- Differential cross section in the lab. frame ($m_1 = 0$):

$$\frac{d\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 \left|M_{fi}\right|^2 \Leftrightarrow \frac{d\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1(1-\cos\theta)}\right)^2 \left|M_{fi}\right|^2 \qquad \textbf{(22)}$$

- Differential cross section in the lab. frame ($m_1 \neq 0$):

$$\frac{d\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\frac{1}{|\overrightarrow{p_1}|m_1}\frac{|\overrightarrow{p_3}|^2}{|\overrightarrow{p_3}|(E_1+m_2)-E_3|\overrightarrow{p_1}|\cos\theta}\left|M_{fi}\right|^2 \qquad \textbf{(23)}$$

$$(E_1+m_2) = \sqrt{|\overrightarrow{p_3}|^2+m_3^2}+\sqrt{|\overrightarrow{p_1}|^2+|\overrightarrow{p_3}|^2-2|\overrightarrow{p_1}||\overrightarrow{p_3}|\cos\theta+m_4^2}$$

Examples: relativistic Rutherford cross section

- Need to get a differential cross section in the lab frame with a massless incident particle $M \gg E_1$ (Eq. 22)

$$\frac{d\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 M^2} \left|M_{fi}\right|^2 \qquad \mathbf{(24)}$$

- using

$$\left|M_{fi}\right|^2 = (2E_1 \cdot 2M \cdot 2E_3 \cdot 2M)\left|T_{fi}\right|^2 \qquad \mathbf{(25)}$$

$$\Longrightarrow \frac{d\sigma}{\mathrm{d}\Omega} = \frac{16M^2 E_1 E_3}{64\pi^2 M^2} \left|T_{fi}\right|^2 = \frac{E^2}{(2\pi)^2} \left|T_{fi}\right|^2 \qquad \mathbf{(26)}$$

- Hamiltonian for a Coulomb potential: $\hat{H} = e\phi(x)$

- Initial and final state wave function are plane waves: $|\Psi_i\rangle = e^{-ip_1 \cdot x}, \left|\Psi_f\right\rangle = e^{-ip_3 \cdot x}$

- Matrix element T_{fi}:

$$T_{fi} = \left\langle \Psi_f \middle| \hat{H} \middle| \Psi_i \right\rangle = \int e^{ip_3 \cdot x}\, e\phi(x) e^{-ip_1 \cdot x} d^3 x \qquad \mathbf{(27)}$$

Examples: relativistic Rutherford cross section

- Define momentum transfer as $q = p_3 - p_1$

$$T_{fi} = \langle \Psi_f | \widehat{H} | \Psi_i \rangle = e \int \phi(x) e^{iq\cdot x} d^3x \qquad \mathbf{(28)}$$

- use

$$e^{iq\cdot x} = -\frac{1}{|q|^2} \cdot \nabla^2 e^{-iq\cdot x} \text{ and } \int (u\nabla^2 v - v\nabla^2 u)\, d^3x = 0 \text{ (Green's theorem)} \qquad \mathbf{(29)}$$

$$T_{fi} = -\frac{e}{|q|^2} \int \left(\nabla^2 \phi(x)\right) e^{-iq\cdot x}\, d^3x \qquad \mathbf{(30)}$$

- From Poisson equation: $\nabla^2 \phi(x) = -\rho(x)$, where $\rho(x) = Zef(x)$ is a static potential with a normalisation condition $\int f(x) d^3x = 1$

$$T_{fi} = \frac{4\pi\alpha Z}{|q|^2} \int f(x) e^{-iq\cdot x}\, d^3x \qquad \mathbf{(31)}$$

Examples: relativistic Rutherford cross section

- Definition: $F(q) = \int f(x) e^{-iq\cdot x}\, d^3x$ is the Fourier transformation of the charge function $f(x)$, called also a Form Factor (FF) of the charge distribution
- The FF contains all the information about the spatial distribution of the charge of the studied object
- In our case we replace it with a delta function $F(q) = 1$
- This gives for the matrix element simply:

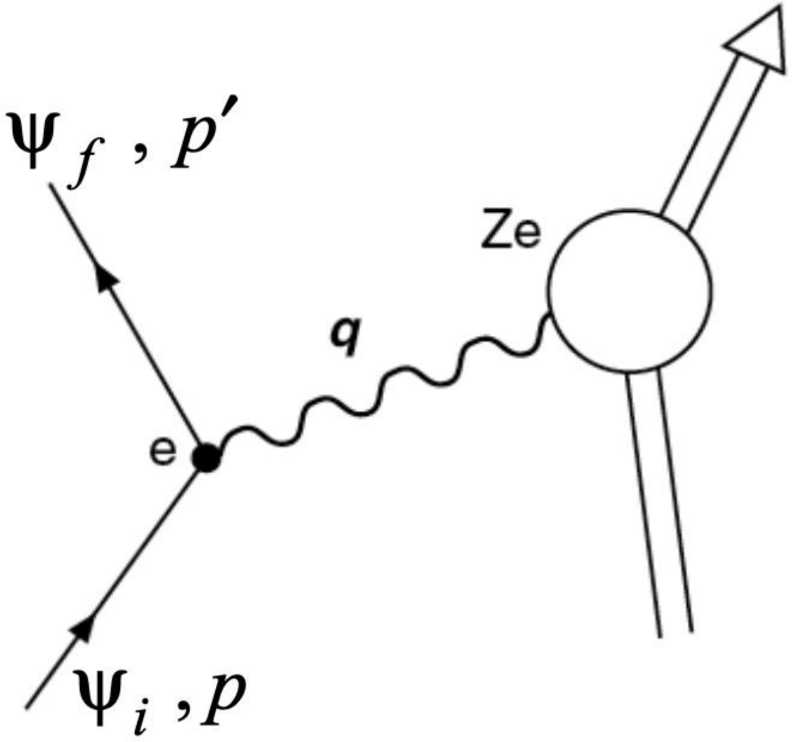
$$T_{fi} = \frac{4\pi\alpha Z}{|q|^2} \qquad \mathbf{(32)}$$

- So the Rutherford cross section becomes:

$$\frac{d\sigma}{\mathrm{d}\Omega} = \frac{E^2}{(2\pi)^2}\left|\frac{4\pi\alpha Z}{|q|^2}\right|^2 = \frac{4Z^2\alpha^2E^2}{|q|^4} \qquad \mathbf{(33)}$$

Examples: relativistic Rutherford cross section

- Quantum Field theory: the electron interacts with a nucleus (charge = Ze) via the exchange of a photon
  - photon momentum: $q = p - p'$
  - de Broglie wavelength: $\lambda = 1/|q|$
- If λ is large, the internal structure of the nucleus can not be resolved and can be considered as a point-like object (that is what we assumed so far in our calculations)



Particle accelerators: motivations

- Accelerators serve as "microscopes"
  - $|q| = 1\ \text{GeV} \implies \lambda \approx 1.2 \times 10^{-15}\ \text{m}$ – size of a proton
  - $|q| = 10^3\ \text{GeV} \implies \lambda \approx 1.2 \times 10^{-18}\ \text{m}$ – size of a proton substructure (e.g. quarks)
  - $\implies$ accelerators allow us to look for the substructure of particles
- Types of operation modes:
  - fixed target of mass m, beam with energy E: $\sqrt{s} \approx \sqrt{2mE}$
  - collider with two beams of energy E: $\sqrt{s} = 2E$
- Example: collider with two 22 GeV (1 TeV) beams gives the same center-of-mass energy as a fixed target with a beam of 1 TeV (10^3 TeV)
- When looking for massive particles produced in the interactions, aim for the highest energy possible and the collider mode is the more appropriate one

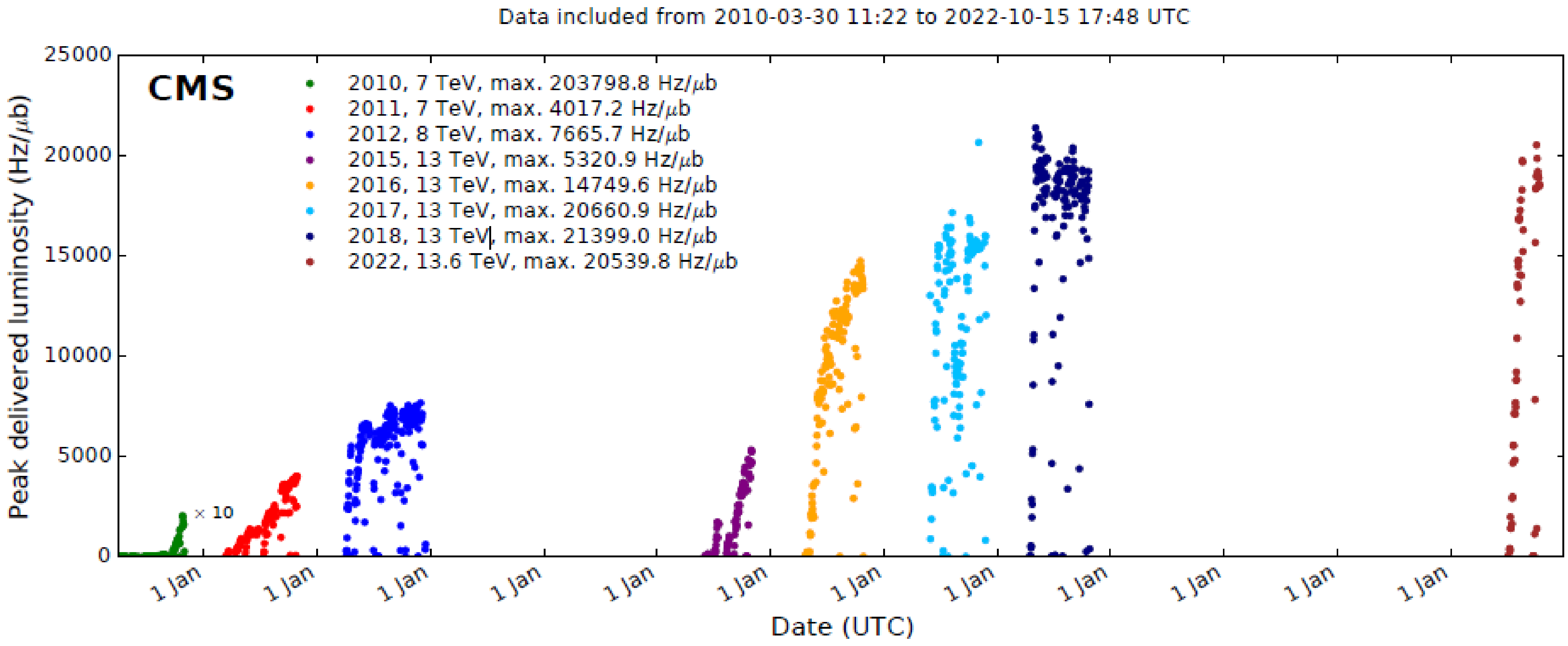
Particle accelerators: luminosity

- Luminosity ($\mathcal{L}$) is the exposure of the target (beam) to scattering (collision) per unit time and unit area
- **Fixed target experiments**
  - $\mathcal{L} =$ flux $\times$ number of scattering centers $= \Phi_a \times N_b = n_a \times v_a \times N_b$
- **Colliding beams**
  - $\mathcal{L} = \frac{n_a \times n_b}{A} bf$, where $A = 4\pi\sigma_x\sigma_y$
  - here n_i is the number of particles per bunch, b is the number of bunches, f is the frequency of the orbit
  - typical beam size of the LHC $\sigma_x \approx \sigma_y \approx 15\mu$m
- The number of produced events for a process with cross section σ is:

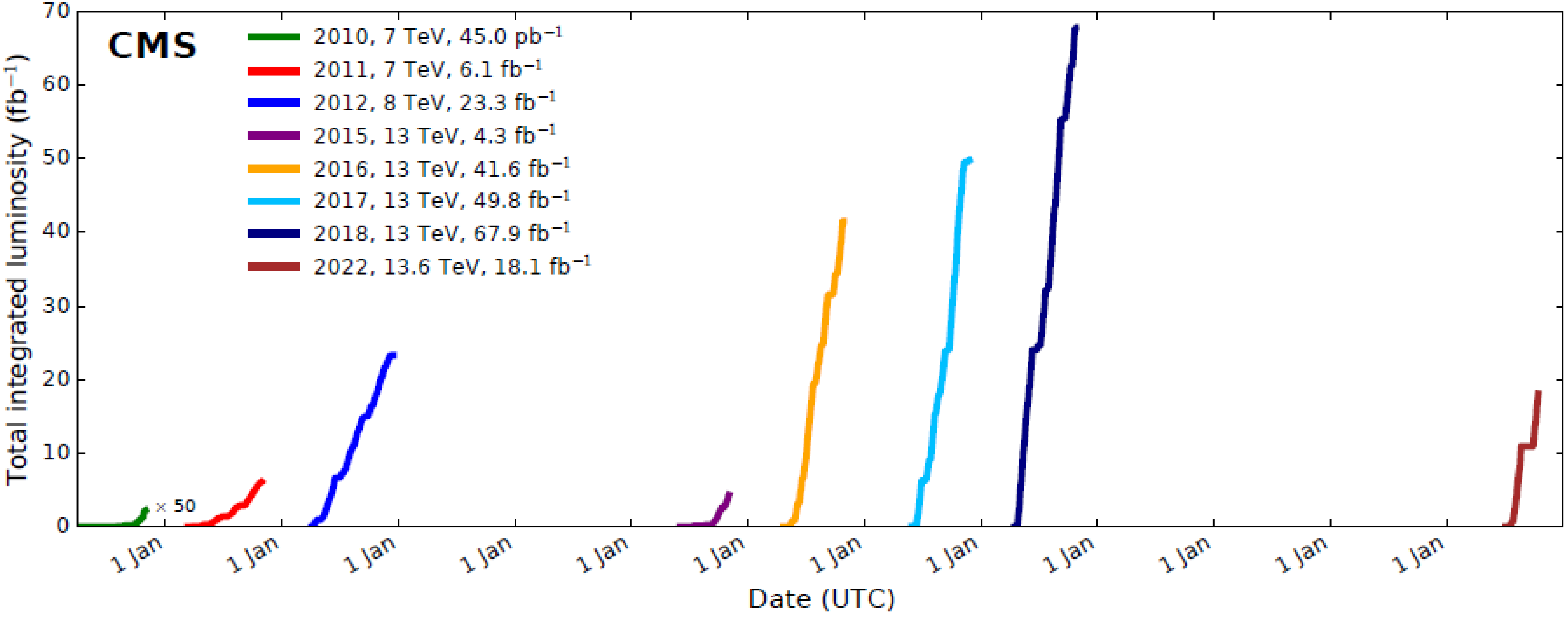
$$N = \mathcal{L}_{\text{int}} \times \sigma, \qquad \mathcal{L}_{\text{int}} = \int \mathcal{L}\, dt$$

LHC: instantaneous luminosity $\mathcal{L}$

Peak instantaneous luminosity :

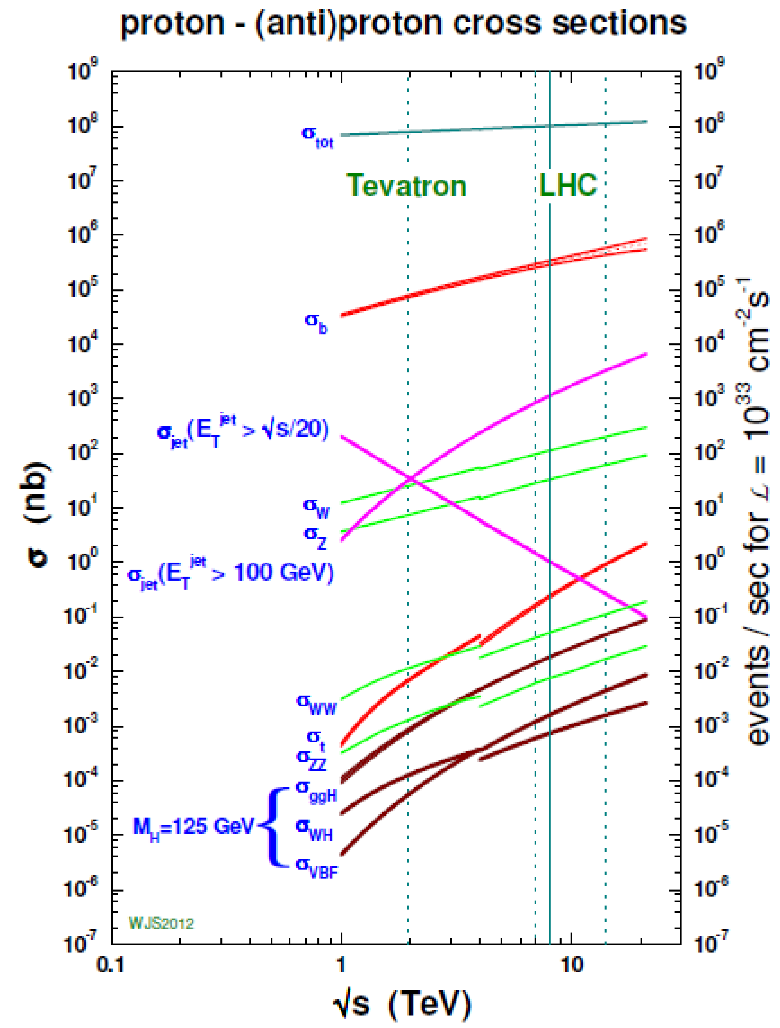


LHC: integrated luminosity $\mathcal{L}_{\mathbf{int}}$



LHC: example production cross section

- $N_{pp} = \mathcal{L}_{\text{int}} \sigma_{\text{p}}^{\text{inel}} = 20 \cdot 100 \cdot 10^{12} \left(\text{with } \sigma_{\text{p}}^{\text{inel}} = 100mb\right)$ cross section at 13.6 TeV
  - W boson (leptonic decays): 63000 pb
  - Z boson: $Z \rightarrow \mu\mu$: 2103 pb
  - $t\bar{t}$: 920 pb
  - gluon fusion Higgs production: 53 pb
  - $t\bar{t}H$: 600 fb
  - HH: 34 fb



- How many given interactions did LHC produce in 2022?
  - assume $\mathcal{L} = 20\text{fb}^{-1}$?

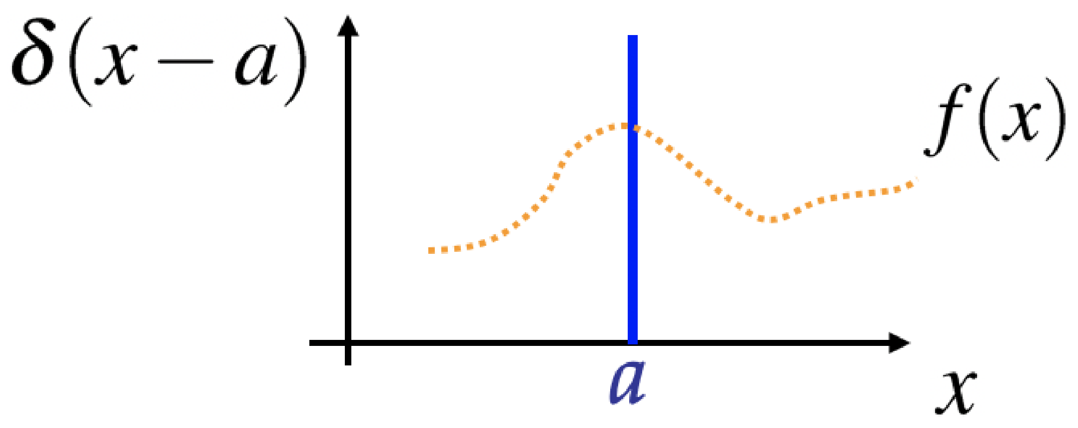
Summary of Lecture 4

Main learning outcomes

- How to compute 2-body decay rates using Fermi's golden rule
- How to deal with kinematics of particle decays and cross sections
- The fundamental particle physics is in the matrix element
- The above equations are the basis for all calculations that follow

Additional slides: Dirac δ –function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: "infinitely narrow spike of unit area"



$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

- Any function with the above properties can represent $\delta(x)$, e.g.:

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$

Infinitesimally narrow Gaussian

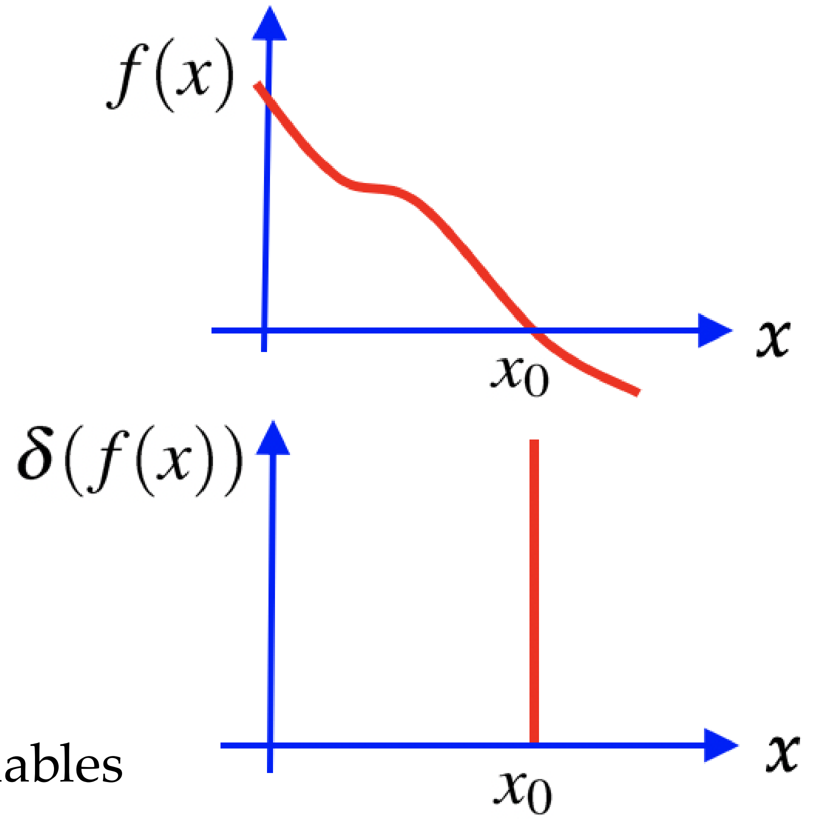
Additional slides: Dirac δ −function of a function

- An expression for the δ −function of a function $\delta\big(f(x)\big)$:
  - start from the definition of a δ −function:

$$\int_{y_1}^{y_2} \delta(y)dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- Now express in terms of $y = f(x)$, where $f(x_0) = 0$ and change variables

$$\int_{x_1}^{x_2} \delta\big(f(x)\big)\frac{df}{dx}dx = \begin{cases} 1 & if\ x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



Additional slides: Dirac δ −function of a function

- From the properties of a δ −function (i.e. only non-zero at x_0)

$$\left|\frac{df}{dx}\right| \int_{x_1}^{x_2} \delta(f(x))dx = \begin{cases} 1 & if\ x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- Rearranging and expressing RHS as a δ −function

$$\int_{x_1}^{x_2} \delta(f(x))dx = \frac{1}{\left|\frac{df}{dx}\right|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)dx \Longrightarrow \delta(f(x)) = \left|\frac{df}{dx}\right|_{x_0}^{-1} \delta(x - x_0)$$